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"QUANTITY OF MATTER" IN DYNAMICS.[1]

By L. M. HOSKINS, Stanford University.

I. INTRODUCTORY.

1. *Different Views Regarding Mass.*—In explaining the significance of the quantity commonly called *mass* the term "quantity of matter" has been freely used by many writers on the laws of motion, including such high authorities as Newton, Maxwell, Kelvin, Tait and Clifford. These writers have assumed, either explicitly or tacitly, that the words have a meaning apart from dynamical laws. From this point of view the second Newtonian law (or "law of acceleration") is a statement of the way in which the *acceleration* of a body depends upon the two factors, (1) the *force* acting on the body and (2) the *quantity of matter* in the body. There are, however, those who dissent from this view, maintaining that the word mass as used in dynamics is fully defined by the law of acceleration itself; that the statement that "the mass of a body is a measure of its quantity of matter" contributes nothing to the definition. Because of these differing views it seems profitable to consider somewhat carefully what is really involved in the conception of quantity of matter.

In this study we may ask first what meaning, if any, may properly be attached to the words quantity of matter independently of the laws of dynamics, and secondly, whether the use of these words contributes anything to the definition of the quantity usually called mass in the formulation of dynamical principles.

2. *Postulates Regarding Acceleration and Force.*—To answer the second question requires an analysis of the logical import of the laws of motion; and such an analysis, if exhaustive, must involve not only the meaning of mass and force but also the question of bases of reference for estimating acceleration. The present object does not, however, require an exhaustive analysis of these questions. As regards acceleration, we shall assume as granted either the Newtonian view of absolute motion or the view that, at any rate, there exists a base for which the Newtonian laws are true.[2] As regards force, we shall treat separately two assumptions—(1) that force may be accepted as an exactly measurable magnitude independently of the laws of motion, and (2) that the definition of force as a measurable magnitude is contained in those laws; in accordance with these two assumptions parallel sets of propositions will be given, designed to illustrate the significance of "quantity of matter" in the law of acceleration.

II. PRELIMINARY QUANTITATIVE NOTIONS REGARDING MATTER.

3. *Definition of Body.*—We regard matter as consisting of individual parts which are indestructible; and we define a body as *any definite connected portion of matter.*

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[2] Such a base may be called a Newtonian base. (See W. H. Macaulay, *Mathematical Gazette*, October, 1900; *Bull. Am. Math. Soc.*, July, 1897, and April, 1898.)

Explanations of the laws of dynamics are often stated as applying to "particles" of matter or "material points." A more practical presentation must, however, deal with aggregates of matter of finite size. We shall here mean by a body any connected aggregate of matter; and when reference is made to the acceleration of a body, it is to be understood that all parts of the body have equal accelerations, *i. e.*, that the motion is a translation.

4. *Common Notion About Quantity of Matter.*—That a quantitative view of matter is one of the common notions associated with every-day experience is evident on a very little reflection. The statement that one of two gold coins contains twice as much of that particular material as the other is certainly not meaningless, quite apart from any consideration of inertia, or of weight, or of volume, or of value; although a precise explanation of the statement would be as difficult as would be a precise explanation of the statement that one body has twice the bulk of another. The notion of quantity, associated with particular kinds of matter, is in fact employed daily by the many persons who are engaged in buying and selling useful commodities. Whatever method may be used for the actual comparison of quantities, the object sought is to determine definite amounts of certain kinds of matter, the utility of which is not generally due to their inertia, or weight, or bulk.

5. *Definite Quantitative Notions Applicable to Matter.*—There are, moreover, certain very definite quantitative comparisons which are applicable to matter, without restriction to bodies of the same homogeneous substance, and without appeal to dynamical principles or to particular physical laws such as the law of gravitation. We recognize the applicability to matter of the notions that

(I) *The whole is greater than any part, and the whole is equal to the sum of its parts.*

And thus, quite independently of any kinetic considerations, we accept the truth of the following propositions:

(II) *If a body A be divided into two bodies B and C, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

(III) *If any two distinct bodies B and C be combined into a single body A, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

III. SIGNIFICANCE OF QUANTITY OF MATTER IN THE LAW OF ACCELERATION, ASSUMING FORCE TO BE DEFINED NON-KINETICALLY.

6. *First Assumption Regarding Force.*—We now assume that force is a quantity which can be exactly measured, or at least defined as an exactly measurable magnitude, without any appeal to the laws of motion. Thus we may accept as satisfactory a method of comparing force-magnitudes based upon the deformations of springs or of any elastic bodies. It will be seen, however, that in the first set of illustrative propositions given below the only requirement regarding the measurement of forces is that it is possible to apply equal forces to any different

bodies; while in the second set it is assumed possible to apply forces of any desired relative magnitudes to any bodies at pleasure.

7. *The Effects of Equal Forces Applied to Different Bodies.*—To illustrate the way in which the notion of quantity of matter enters into our interpretation of the law of acceleration, we may state a number of propositions referring to suppositious cases in which equal forces act upon different bodies.

(*a*) If a force be applied to a body B, and an equal force be afterward applied to a body A formed by adding matter to B, the acceleration of A will be less than that of B.

(*b*) If, when equal forces are applied to two distinct bodies A and B, the acceleration of A is less than that of B, then by the removal of matter from A there may be produced a body A' such that equal forces applied to A' and B will cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces applied to A and B' will cause equal accelerations.

(*c*) If two distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration half as great will be caused by an equal force acting upon the body formed by combining the two.

(*d*) If any number of distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration one nth as great will be caused by an equal force acting upon a body formed by combining n of the given bodies.

(*e*) If two distinct bodies, when acted upon by equal forces, have accelerations a' and a'', then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a''$.

(*f*) If any number of distinct bodies, when acted upon by equal forces, have accelerations $a', a'', \cdots$, then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a'' + \cdots$. (It will be noticed that this case really includes the five preceding.)

These propositions are consequences of the interpretation we put upon the following principle, which is a part of the law of acceleration:

(*A*) *Different bodies acted upon by equal forces have accelerations inversely proportional to their masses.*

A part of the import of proposition (*A*) undoubtedly consists in giving precision to the definition of mass; but if the meaning of mass were wholly contained in it, not one of the propositions (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) would necessarily follow from it. The reason we accept these as consequences of (*A*) is that we have, independently of (*A*), a conception of mass as a measure of quantity of matter and as satisfying propositions (I), (II) and (III).

8. *Case of Different Bodies Having Equal Accelerations.*—The part played by the quantitative notion of matter in our interpretation of the second law of motion is evident also in cases in which different bodies are supposed to be acted upon by forces such as to cause equal accelerations. This is brought out by the following propositions, which will be seen to have a correspondence to the propositions (*a*) $\cdots$ (*f*) given above.

(a') If a body A be formed by adding matter to a body B, the force required to give A any certain acceleration is greater than that previously required to give B an equal acceleration.

(b') If A and B are two distinct bodies such that a greater force must be applied to A than to B to cause equal accelerations, then by the removal of matter from A there may be produced a body A' such that equal forces must be applied to A' and B to cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces must be applied to A and B' to cause equal accelerations.

(c') If two distinct bodies have equal accelerations when acted upon by equal forces, then a body formed by combining the two must be acted upon by twice as great a force in order to have an equal acceleration.

(d') If any number of distinct bodies have equal accelerations when acted upon by equal forces, then the body formed by combining any n of them must be acted upon by a force n times as great in order to have an equal acceleration.

(e') If the forces required to give two distinct bodies equal accelerations are F' and F'', then the body formed by combining the two must be acted upon by a force $F' + F''$ in order to have an equal acceleration.

(f') If the forces required to give any number of distinct bodies equal accelerations are F', F'', $\cdots$, then the body formed by combining them must be acted upon by a force $F' + F'' + \cdots$ in order to have an equal acceleration. (This includes the five preceding cases.)

These propositions are accepted as consequences of the following principle, which expresses part of the import of the law of acceleration:

(B) *In order that different bodies may have equal accelerations, they must be acted upon by forces whose magnitudes are proportional directly to the masses of the bodies.*

Assuming that force as a measurable magnitude is defined independently of this principle, a part of its import consists in giving precision to the meaning of mass; but if the whole meaning of mass were included in (B) no one of propositions (a') $\cdots$ (f') would necessarily follow from it. We accept these as consequences of (B) because we have, independently of (B), the conception that mass is a measure of quantity of matter and satisfies propositions (I), (II) and (III).

9. *Effects of Different Forces Applied at Different Times to the Same Body.*—Besides principles (A) and (B), each of which expresses a part of the import of the law of acceleration, there is a third principle of equal interest which may be stated as follows:

(C) *Different forces applied at different times to the same body cause accelerations having the same ratios as the magnitudes of the forces.*

This is not independent of (A) and (B); in fact, careful reflection shows that any one of the three propositions (A), (B) and (C) may be inferred from the other two.

10. *General Statement of Law of Acceleration.*—The general principle of which (A), (B) and (C) are particular cases may be stated as follows:

(*D*) *Any bodies acted upon by any forces will have accelerations proportional directly to the force-magnitudes and inversely to the masses* (*quantities of matter*) *of the bodies.*

Principle (*D*) may be called the general law of acceleration; it is equivalent to Newton's second law of motion, so far as this refers to forces acting singly.

IV. SIGNIFICANCE OF QUANTITY OF MATTER WHEN FORCE IS DEFINED KINETICALLY.

11. *Second View of Force.*—It is generally recognized that, while the measurement of forces by their effects in deforming elastic bodies (as by the spring balance) is of very great practical use, the exact comparison of force-magnitudes for the purposes of dynamics must make use of the law of acceleration. If this view is adopted a different form must be given to the explanation of the rôle of mass or quantity of matter in the laws of motion. To this we proceed.

12. *Formulation of the Laws of Motion.*—A rigorous logical analysis shows that the second and third laws of Newton (the law of acceleration and the law of action and reaction) are not independent principles. It will be seen, however, that the import of both is covered by the following propositions:

(1) *The acceleration of a body is always due to the influence of other bodies.*

(2) *If the acceleration of a body is due to the influence of more than one other body, it is equal to the vector sum of components, each due to some one of the bodies.*

(3) *If A and B are any two individual portions of matter, the acceleration of A due to B and that of B due to A are always oppositely directed, and their magnitudes are in a constant ratio.*

Definition.—The term "acceleration-ratio of A with respect to B" will be used to denote the ratio:

$$(\text{acceleration of } A \text{ due to } B)/(\text{acceleration of } B \text{ due to } A).$$

(4) *If A, B and C are any three distinct bodies, the acceleration-ratio of A with respect to B is equal to the acceleration-ratio of A with respect to C divided by the acceleration-ratio of B with respect to C.*[1]

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[1] In connection with (3) it should be added that if A and B are "particles,"—i. e., if their dimensions are vanishingly small in comparison with their distance apart,—their mutually-caused accelerations are directed along the line AB.

The "law of composition," expressed by (2), is sometimes held to mean that the acceleration of a body A due to a body B is not influenced by any third body C; in other words, that if a' is the acceleration of A due to B when C is absent, and a'' the acceleration of A due to C when B is absent, then when both B and C are present the acceleration of A will be the vector sum of a' and a''. This cannot be accepted as a valid interpretation of the law of composition; all that is implied by this law is that the acceleration of every particle can at every instant be resolved into components regarded as individually "due to" other particles in the sense that they satisfy (2), (3) and (4). The supposition that the acceleration of A due to B may be changed by the presence of C is entirely consistent with these laws. (See K. Pearson, "The Grammar of Science," second edition, page 317.)

The import of the laws is, in fact, summed up in the statement that the total momentum of a system of bodies is not changed by the action of these bodies upon one another (total momentum being understood in its broadest sense as a localized vector quantity). The definition of momentum must of course depend upon that of mass given in 13.

13. *Definition of Mass.*—These principles imply that *it is possible to assign to all bodies individual constant numbers such that the acceleration-ratio of any two bodies is the inverse ratio of their assigned constants.* Constants satisfying this condition are called the *masses* of the bodies. The actual values of the mass-constants are fixed as soon as any one of them is assigned; this is equivalent to choosing a unit of mass.[1]

14. *Mass as Quantity of Matter.*—That there is any relation between the quantity thus defined as mass and matter regarded as a quantity is not immediately apparent from principles (1), (2), (3) and (4). That there is such a relation, and that mass as just defined is an appropriate measure of quantity of matter, is illustrated by the following propositions referring to suppositious cases. It will be seen that there is a correspondence between these propositions and those above designated as $(a) \cdots (f)$.

(a'') If a body A be formed by adding matter to a body B, then the acceleration-ratio of A with respect to any third body D will be less than that of B with respect to D.

(b'') If A and B are two distinct bodies such that the acceleration-ratio of A with respect to a third body is less than that of B with respect to that body, then by the removal of matter from A there may be produced a body A' such that the acceleration-ratios of A' and B with respect to any third body are equal; and by the addition of matter to B there may be produced a body B' such that the acceleration-ratios of A and B' with respect to any third body are equal.

(c'') If the acceleration-ratios of two bodies B and C with respect to a third body D have equal values r, then if B and C be combined into a single body A the acceleration-ratio of A with respect to D will be $r/2$.

(d'') If there be any number of bodies having equal acceleration-ratios r with respect to a body D, then if any n of them be combined into a single body its acceleration-ratio with respect to D will be r/n.

(e'') If two distinct bodies have acceleration-ratios r', r'' with respect to another body D, then the body formed by combining the two will have with respect to D an acceleration-ratio r such that $1/r = 1/r' + 1/r''$.

(f'') If any number of distinct bodies have, with respect to a body D, the acceleration-ratios r', r'', $\cdots$, then the body formed by combining them will have, with respect to D, an acceleration-ratio r such that $1/r = 1/r' + 1/r'' + \cdots$. (This includes the five preceding cases.)

These propositions are consequences of the principle that

(E) *The acceleration-ratios of any bodies* $A, B, C, \cdots$, *with respect to any one other body are in the inverse ratios of the quantities of matter of* $A, B, C, \cdots$.

A part of the import of (E) is a precise *definition* of quantity of matter; but unless we had, independently of (E), a quantitative conception of matter satisfying propositions (I), (II) and (III), we should not recognize that (a''), $\cdots$ (f'') are consequences of (E); and we certainly should not recognize that they are consequences of (1), (2), (3) and (4).

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[1] The above method of formulating the laws of motion and of defining mass is substantially that which has been used by a number of writers, among whom may be mentioned E. MACH, K. PEARSON, G. KIRCHHOFF, P. APPELL, A. E. H. LOVE, W. H. MACAULAY. Some of these authors have explicitly recognized that the conception of mass which we actually use includes the notion of a quantitative measure of matter satisfying propositions (I), (II) and (III).

Proposition (E) shows that the quantity above defined as *mass* is an appropriate measure of *quantity of matter*.

15. *Definition of Force.*—From the present point of view the import of the Newtonian laws is completely covered by principles (1), (2), (3), (4) and (E); and it will be noticed that the word force is not used in the statement of these propositions. A definition of force may, however, be associated with them. Thus (1) and (2) suggest the following:

A force is an action exerted by one body upon another, the effect of which is to give the latter body an acceleration-component.

With this definition it is seen that (2) expresses the law of vector composition of forces.

An exact quantitative definition of force, consistent with the above principles, may be stated as follows:

A force is an action exerted by one body upon another, measured by the product of the acceleration produced and the mass of the body acted upon.

With this definition, and that of mass given above, principle (3) is seen to be equivalent to the law of action and reaction in its ordinary form:

Whenever one body exerts a force upon another, the second exerts an equal and opposite force upon the first.

V. Mass in the Law of Gravitation.

16. *Laws of Dynamics Independent of Law of Gravitation.*—Because the weights of bodies in the same locality are proportional to their masses, weight[1] plays an important part not only in the practical work of comparing the masses of bodies but in explanations of the principles of dynamics. It must not be forgotten, however, that the laws of dynamics are quite independent of the law of gravitation.

17. *Significance of Mass in the Law of Gravitation.*—The law of gravitation states that every portion of matter exerts an attractive force upon every other portion, and that the magnitude of the attractive force exerted by one particle[2] upon another depends in a definite way upon the masses of the particles and their distance apart. It is of interest to remark that all the questions which have been raised regarding the significance of mass in the law of acceleration might be raised with equal pertinence with regard to the meaning of mass in the law of gravitation. Thus the statement that "two particles which are at equal distances

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[1] The word weight is often used popularly in the sense of quantity of matter, and some writers recommend the general adoption of this usage. There would be no objection to this if it did not lead to the confusing of two distinct things; but it is necessary to use some word to denote the gravitational pull of the earth upon a body, and scientific usage is nearly unanimous in assigning this meaning to the word weight. This usage is followed in the present paper.

[2] It is convenient here to refer to particles rather than to bodies of finite dimensions in order to avoid complexities of statement due to the fact that forces combine by vector addition.

from a third particle are attracted by it with forces proportional to their masses" might be held to be merely a definition of mass. It might be asserted that "the statement that A's mass is three times as great as B's means merely that if A and B are equally distant from C the attractive force of C upon A is three times that of C upon B."

It is not difficult to show, however, that the conception of mass as a measure of quantity of matter is an essential part of the interpretation we put upon the law of gravitation. Thus if, when A and B are equally distant from C, the attraction of C upon A is three times that of C upon B, we say that it is because A has three times as much matter as B; and we infer that A might be divided into three parts each having a mass equal to that of B.

The law of gravitation thus derives an essential part of its significance from a notion of mass which is independent of that law—the notion that mass is a measure of the matter of which bodies are composed and satisfies the fundamental quantitative relations expressed by (I), (II) and (III). At the same time it must be recognized that the law itself gives precision to the definition and furnishes a method of comparing quantities of matter with exactness.

18. *Agreement of Two Methods of Estimating Quantity of Matter.*—It has been seen that the law of acceleration and the law of gravitation furnish two independent methods of applying exact quantitative measurements to matter. That the results of the two methods are in apparently exact agreement is a matter of experimental knowledge rather than of *a priori* necessity. If the fact were otherwise, weighing would not be an available method of comparing the masses of bodies for the purposes of dynamics.

GEOMETRICAL AND OTHER ILLUSTRATIONS OF INDETERMINATE FORMS.

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In the AMERICAN MATHEMATICAL MONTHLY, Vol. III, 1893, pp. 207–8, there is given by B. F. Finkel a geometrical proof that $0 \times \infty$ is indeterminate. In the present paper additional illustrations, geometrical and otherwise, are given of this and other indeterminate forms.

I. The area generated by a segment of a straight line AB (Fig. 1) revolving about an axis CD in its plane is given by the formula:

$$\text{Area } AB = 2\pi \cdot AB \cdot MO = 2\pi \cdot CD \cdot MR,$$

where CD is the projection of the segment AB upon the line CD, M is the middle point of AB, O is the center of the segment CD, and MR is perpendicular to AB. Let AB revolve about M and become perpendicular to CD. Then MR becomes parallel to CD and $CD \cdot MR$ becomes $0 \times \infty$.